Bézout's Theorem — To Infinity and Beyond

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Abstract

Algebraic geometry lives in the land of polynomials and their zero sets. Sometimes, you'd like to know what happens when you've got multiple polynomials floating around. How do they interact? In this talk, I'm going to state a theorem and then ***dis***prove it, five or six times in a row, and each time we take one more step out of Plato's cave. Along the way, we'll go from high school math to undergrad to grad school to tools of active research, but throughout, this talk should be beginner-friendly and accessible to a wide audience. The title is a pun in (at least) two ways.

§1 High school

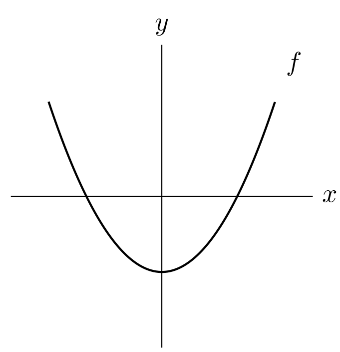
Okay, I'll admit it's been a long time since I've been out of high school, but basically the only thing of significance I can remember doing is finding roots and other such "solve-for-x" type problems. For instance:

What are the roots of $f = x^2 - 1$? Of course, we all know what to do here: set f equal to 0 and factor. We get

$$(x-1)(x+1) = 0$$

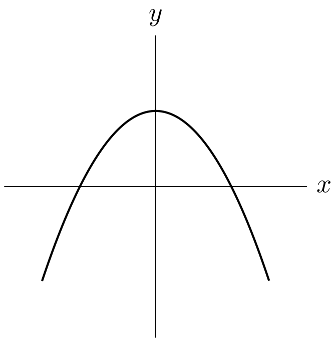
which has solutions $x = 1, -1$.

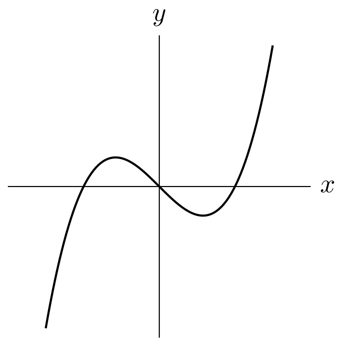
It's worth mentioning something we all take for granted, which is that we don't have to do algebra if we don't want to: you can just graph f and visually *see* the roots.

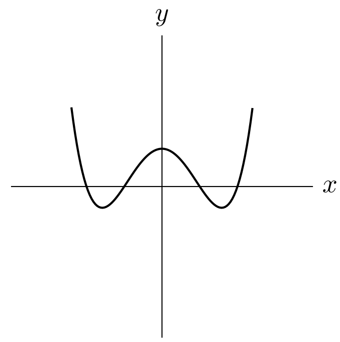


This is (historically) nontrivial: it wasn't until Descartes in the 1600s that anyone even had the thought that you could connect the algebra to a geometric picture of plotting inputs and outputs. It's a remarkable idea and when I try to imagine myself ignorant to the Cartesian plane, I don't think I could've invented it!

But now we're off to the races. You can hand me all sorts of quadratics, or for that matter, cubics, quartics, quintics, etc, and I can quickly and easily find the roots by inspection.



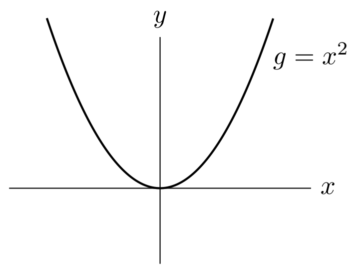




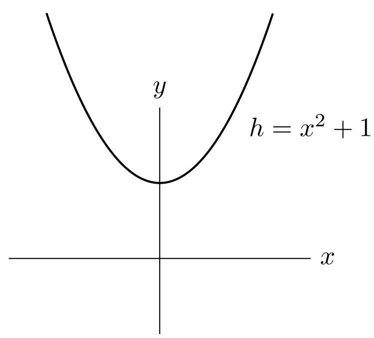
And given enough examples you come to the following lovely theorem:

Theorem 1 (Fundamental Theorem of Algebra)**.** *If* $\deg f = n$*, then* f *has* n *roots.*

But of course we're mathematicians, so we don't take examples as proof, and we set out to prove this. We hem and we haw and get stuck, and then some jerk (maybe one of those Bernoullis) hits us with a:



Or worse, a:



And of course the problem is not exclusive to quadratics. So now you're at an impasse. You can take the coward's way out:

Theorem 2 (Fundamental Theorem of Algebra)**.** *If* $\deg f = n$*, then* f *has at most* n *roots.*

Or you can actually fix things. Of course, we know the fix, but again, let's put ourselves in the historical context. It's Late Middle Ages / Early Modern Era, and you know how to solve quadratics. You realize that admitting a solution $x = i$ to $x^2 + 1 = 0$ suffices to give a method for finding the roots of *any* quadratic. But even cubics — just cubics, not all polynomials! — are seemingly another beast entirely. You have Tartaglia and Cardano legitimately beefing over secret methods to solve cubics, and the general cubic formula wasn't yet widely known. At that time, it was totally conceivable that, just like admitting i, you'd have to admit even more new types of numbers to solve cubics, quartics, etc. Why on earth would i alone, a quadratic extension of $\mathbf{R}$, be enough?

Yet we know that remarkably, it is!

Theorem 3 (Fundamental Theorem of Algebra)**.** *If* $\deg f = n$*, then* f *has exactly* n *roots, when you work over* $\mathbf{C} := \mathbf{R}(i)$ *and you count with multiplicity.*

And this is a thing you can prove, because it's actually true! (I guess you should worry about $f = 0$ for half a second though.)

§2 Advancing towards Bézout

But of course our perspective should not be so quaint. Let's restate what it means to find roots: ultimately, you have a polynomial f and you're trying to find the intersection of $y = f(x)$ with $y = 0$.

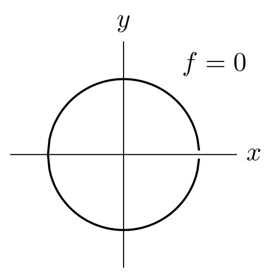
Notice that I could think about $y = f(x)$ instead as a two variable polynomial set equal to 0. An example makes this clear:

$$y = x^2 + x$$
$$y - x^2 - x = 0.$$

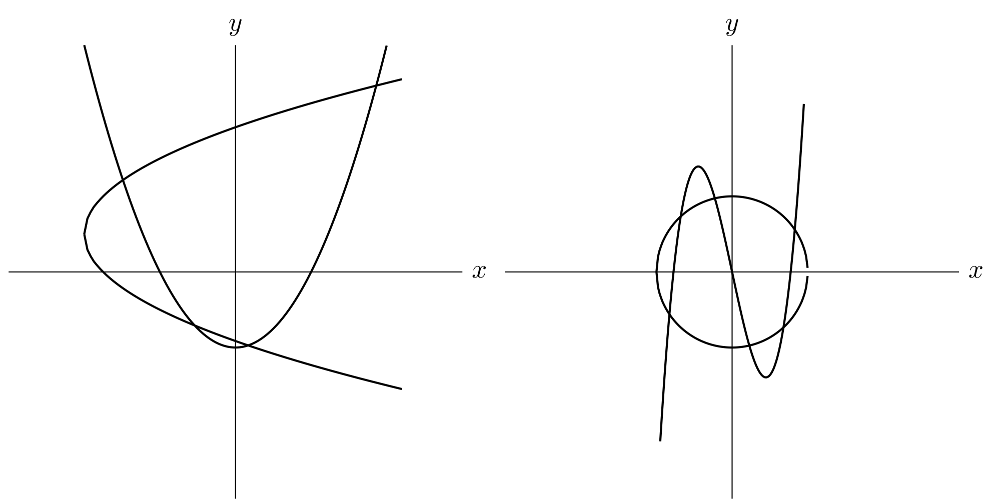
This means we can leave the realm of baby's first polynomials and think about honest multivariate stuff, as long as we set it equal to 0. So maybe you could consider

$$f = x^2 + y^2 - 1$$

whose zero set is the unit circle:



Moreover, since finding roots is intersecting a polynomial with another one (specifically the zero polynomial) you could ask for the intersections not with the x-axis $y = 0$ but with other polynomials.



So now you're graphing all kinds of $f(x, y) = 0$ and looking at how they intersect with each other. And then you do enough examples and cook up another lovely theorem:[1]

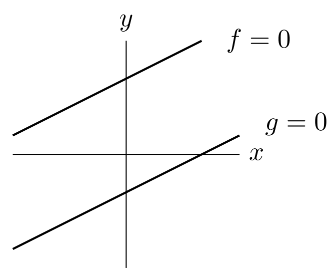
Theorem 4 (Bézout's Theorem)**.** *If* $\deg f = m$ *and* $\deg g = n$*, then* $\{f = 0\} \cap \{g = 0\}$ *has* $m \cdot n$ *points.*

Except! We've been burned before, so we're cautious this time. Notice that Bézout's Theorem as we've stated it right now can't possibly be right, because a subcase of this is the Fundamental Theorem of Algebra: let $f = y - F(x)$ for some F and let $g = y$. Finding $\{y - F(x) = 0\} \cap \{y = 0\}$ is just finding roots of $y = F(x)$. So we probably need to tweak this. We leverage our hard work from earlier and:

[1] Oh, and I guess you rename yourself Étienne Bézout too.

Theorem 5 (Bézout's Theorem)**.** *If* $\deg f = m$ *and* $\deg g = n$*, then* $\{f = 0\} \cap \{g = 0\}$ *has* $m \cdot n$ *points, counting multiplicity, over* $\mathbf{C}$*.*

Haha! We wipe our hands, pat ourselves on the back, and go home considering it a day well spent. Except!



Oh no!

This is a little bit more devastating. I've given you two degree 1 polynomials, which should intersect at $1 \cdot 1 = 1$ point, but that'll never happen, even if I'm in $\mathbf{C}$. Multiplicity doesn't save us either. I mean, if you don't believe the picture, just look for yourself at the algebra:

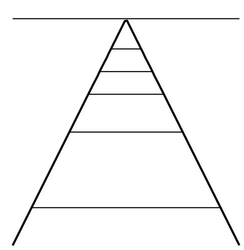
$$\begin{cases} y = 2x + 2 \\ y = 2x - 1 \end{cases}$$

$$\begin{aligned} 2x + 2 &= 2x - 1 \\ 2 &= -1 \end{aligned}$$

Whomp whomp.

Fortunately though there is a solution. Just as we upgraded $\mathbf{R}$ algebraically to $\mathbf{C}$, we will upgrade our number system, this time *topologically*, from $\mathbf{C}$ to something else. The trick is something well-known to our painter friends.[2]

The prototypical example, and stay with me here, is to think about train tracks. By *necessity*, those have to be parallel lines, hence never intersect (even though Bézout says they should once), but look at how we render them:



Those lines are parallel, but in the way our painters project the image onto the canvas, they "intersect" on the horizon, infinitely far away. Thus we should not restrict ourselves to the plane $\mathbf{C}$, but instead the plane plus a point at ∞. This has several names / constructions, but we will call it $\mathbf{P}$, for projective space. And thus:

Theorem 6 (Bézout's Theorem)**.** *If* $\deg f = m$ *and* $\deg g = n$*, then* $\{f = 0\} \cap \{g = 0\}$ *has* $m \cdot n$ *points, counting multiplicity, over* $\mathbf{P}$*.*

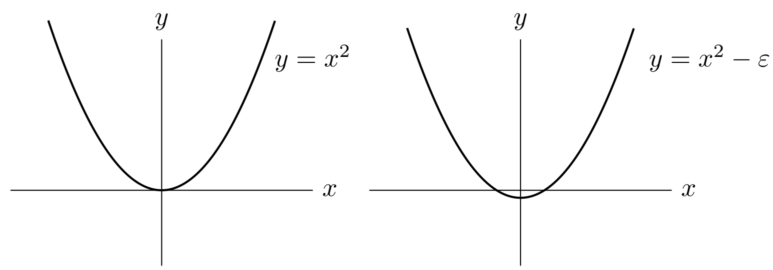
Can we dust off our hands yet? Not quite...

[2] Because again, this is Europe like several hundred years ago and you only had the means to do math / art / etc if you were a well-to-do man.

§3 Quit pulling the wool over our eyes[3]

Okay, we should really back up a bit. What exactly does "counting multiplicity" mean? I mean, we know the vibes, e.g., $y = x^2$ intersects the x-axis with multiplicity 2, but what does that mean? And we should be very careful defining it, because we don't want the answer to be "the number you need to make the Fundamental Theorem of Algebra or Bézout's Theorem work."

There's a topological definition which I will state only for culture, because it ends up being conceptually pretty easy but not very computable. The idea is that if we were to tweak $y = x^2$ only an ε amount, then we would turn the bad intersection point into 2 points.



So this suggests we should define multiplicity via these little tweaks that turn tangent intersections into transverse intersections. You can make this a lot more precise with "cup products" and "linear equivalence classes," but as we're stating it here, it's not very computable because:

1. what happens in higher dimensions? What direction do you "tweak"? It was easy for $y = x^2$ because I can visualize it, but what about a multiple-point on like a 17-variable polynomial?
2. algebraic geometry doesn't concern itself with ε-neighborhoods and analytic statements.

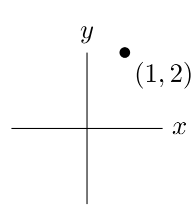
So what is to be done? In fact, the problem with multiplicity isn't the theorem statement but instead our definition of points, and the trick is to stop doing high school level algebra and start doing undergrad algebra.

Up until now we've only been thinking about $\{f = 0\}$ as a collection of points, say in the Euclidean xy-plane. But we're not leveraging a lot of algebra that we could be when we're dealing with polynomials.

A polynomial f like $y - x^2$ is an element of the polynomial ring $\mathbf{C}[x, y]$. By setting $f = 0$, notice that in the polynomial ring that's tantamount to quotienting by (the ideal generated by) f. Certainly in $\mathbf{C}[x, y]/(y - x^2)$, the polynomial $y - x^2$ is equivalent to the zero polynomial. And of course you want to deal with the ideal because if $f = 0$, so too better be $2 \cdot f$, $x \cdot f$, $(x^2 - 3y + 7/3i) \cdot f$, etc. (Also because I don't know how to quotient by something that's not an ideal.)

But now associated to any f that you want to take the zero set of, you get a commutative ring $\mathbf{C}[x, y]/(f)$. This ring will let us upgrade our definition of points: instead of thinking about points in the Euclidean plane, we will think about prime ideals of the ring, and those will be our points.

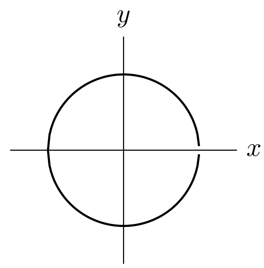
Huh?! Why would this make sense? The answer is because it gives us all the old points we already had, plus new ones that let us do better geometry! The "old points" are the maximal ideals. For a concrete example, consider the ring $\mathbf{C}[x, y]$ which is of course $\mathbf{C}[x, y]/(0)$. So f in this example is the zero polynomial, and thus we're asking for the vanishing of the zero polynomial, but that's the entire xy-plane — the zero function is 0 everywhere. And the maximal ideals of $\mathbf{C}[x, y]$ are ideals of the form $(x - a, y - b)$ for $a, b \in \mathbf{C}$, which correspond exactly to points (a, b) in the xy-plane.



$$(x - 1, y - 2) \subseteq \mathbf{C}[x, y]$$

[3]He says, as he pulls the wool over your eyes *even more*, to try and give an idea of schemes without digging into hardly *any* details.

But you get more points, because there are more prime ideals in $\mathbf{C}[x,y]$ than just the maximal ones. Any irreducible polynomial gives rise to a prime ideal, so you have points like:

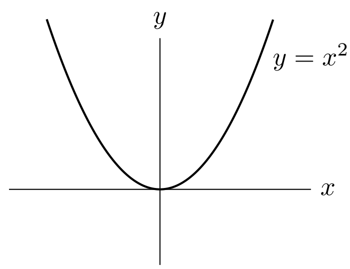


$(f) \subseteq \mathbf{C}[x,y]$. That whole thing is a (single) point.

and also the 0 ideal is a point, not maximal nor an irreducible polynomial (that one's a little funky to picture; ask me after the talk).

When we're not dealing with $\mathbf{C}[x,y]$ and the whole xy-plane, but instead a curve $\{f=0\}$ and the ring $\mathbf{C}[x,y]/(f)$, prime ideals in there are just prime ideals on $\mathbf{C}[x,y]$ that vanish on f. So you get all the old points passing through $\{f=0\}$ (maximal ideals), and also points from irreducible polynomials that evaluate to 0 on every point of $\{f=0\}$.

So now that we know what points should be, how do we use the commutative rings to find intersections? If we want to intersect $y=x^2$ and the x-axis,



then we have the rings $\mathbf{C}[x,y]/(y-x^2)$ and $\mathbf{C}[x,y]/(y)$. Calculating the intersection involves thinking of them as $\mathbf{C}[x,y]$-modules and computing a tensor product:

$$\frac{\mathbf{C}[x,y]}{(y)} \otimes_{\mathbf{C}[x,y]} \frac{\mathbf{C}[x,y]}{(y-x^2)} \cong \frac{\mathbf{C}[x,y]}{(y,y-x^2)} \cong \frac{\mathbf{C}[x,y]}{(x^2,y)} \cong \frac{\mathbf{C}[x]}{(x^2)}.$$

To find the multiplicity here, we calculate the dimension of $\mathbf{C}[x]/(x^2) \cong \mathbf{C} + \mathbf{C}x$ as $\mathbf{C}$-vector space, which is 2. Good — that's what we expected!

Okay, some of this you may be taking on faith, and I'm simplifying to the point of almost being wrong at times, but we have the core idea: we upgrade the space to $\mathbf{P}$, we upgrade the zero sets to schemes, and we upgrade the intersection calculations to tensor product calculations. Now, surely:

Theorem 7 (Bézout's Theorem)**.** *If* $\deg f = m$ *and* $\deg g = n$*, then the corresponding scheme intersection has* $\mathbf{C}$*-dimension* $m \cdot n$ *over* $\mathbf{P}$.

"Please," you beg and plead, "please Eric, please say that *now* we're done."

§4 We aren't done until I say we're done

But what could possibly go wrong? We've seemingly addressed every issue with Bézout's Theorem at this point. There ought not be any issues remaining. But in fact, however, the issue is *twofold*! The first is geometric, and like the parallel lines example, hopefully obvious in hindsight: what about the intersection of a curve and itself?

Now we've really stepped in it. Not only does multiplicity seem like a real pain now, no, even worse than that, the set intersection is too big! We're supposed to be counting discrete sets of points, but a curve intersects itself everywhere — at infinitely many points! And ∞ seems like a stupid answer. It certainly doesn't agree with the expected value of $(\deg f)^2$.

The second problem is a little more esoteric, but if you know a bit about tensor products, you might know what could happen. Tensor products aren't always nicely behaved and can kill a lot of information. In particular, if $M \hookrightarrow M'$ is injective, there's no reason on earth $M \otimes_R N \to M' \otimes_R N$ should be injective.

Here's an easy example to calibrate. Consider the homomorphism $\varphi\colon \mathbf{Z} \to \mathbf{Z}$, $\varphi(x) = 2x$. That's clearly injective because $\ker \varphi = 0$. But if we tensor everything by $\mathbf{Z}/2\mathbf{Z}$, we get the zero map, because $2 \equiv 0 \pmod 2$.

So on the geometry side, we can't consider self-intersections, and on the algebra side, tensor products could kill information, and so our multiplicity counts could be off. These are actually the same problem, because self-tensoring is one of the exact sorts of things that could kill information. The idea is, mirroring our previous calculation,

$$\frac{\mathbf{C}[x,y]}{(f)} \otimes_{\mathbf{C}[x,y]} \frac{\mathbf{C}[x,y]}{(f)} \text{``}\cong\text{''} \frac{\mathbf{C}[x,y]}{(f,f)},$$

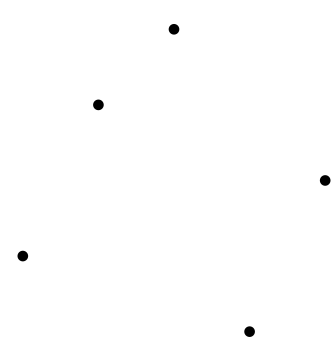
but "quotienting by f twice" is kinda nonsense. If we set $f = 0$ in $\mathbf{C}[x,y]$, you can't exactly do that a second time.

You might be familiar with how one repairs the failure of tensor products to preserve injections. The trick is to *derive* the tensor product, giving you a whole bunch of Tor groups, which is super important but since this talk is aimed at a more elementary audience I won't dive into homological algebra (maybe a later talk!). Let's talk about the geometric problem. The trick is that, while introducing rings was a great idea, we need an even more robust gadget to handle this problem, bringing us to:

§5 Categorical rings[4]

The idea is that we'll expand our notion of what it means to be a ring to "categorical rings" defined via categories. Rather than give you a precise run-down, let's first see an example. While we you should have no idea where we're going or why it would work, one thing to point out before we see these wacky new rings is that all of the rings we're already familiar with should still be examples in this theory. So: how do ordinary rings look like "categorical rings?"

Hand me your favorite ring R. We're going to build a category out of it. The objects of the category will just be the elements of the ring. For the arrows, we always **must** have identity arrows, but we won't admit any others.

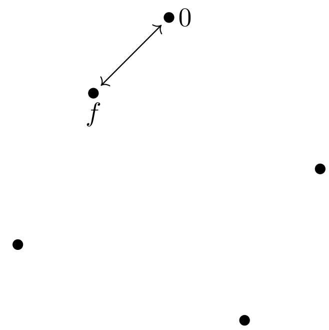


R (the identity arrows are not drawn here)

[4] Actually, this won't be quite right either, but it's good enough vibes for the kind of talk I'm pitching.

Now, to incorporate the ring structure, the addition and multiplication will be functors from pairs of objects to objects.

We can also make sense of (ordinary) quotients in this view. Let's quotient by f and remember we have thought about that as setting $f = 0$. But as you might know, categorically we tend not to ask for things to be equal but instead merely *isomorphic*. So categorically, we shouldn't identify f and 0 as equal, but instead expand the category by introducing a new isomorphism $f \leftrightarrow 0$.



$R/(f)$[5] (the identity arrows are not drawn here)

Notice that this construction makes sense *even if* f is already quotiented. We're expanding the category when we quotient, not killing information, so we can just add a second, different, isomorphism between f and 0. That's a different, bigger category than the one from $R/(f)$, and hence it tells the difference between quotienting once and quotienting twice!

One final question: what's the geometry supposed to be here? What are points?? I don't exactly know what prime ideals of categories are. Fortunately, it's not too bad. The space is actually the exact same as the ordinary rings! That means that even if I quotient by f once or twice or ten thousand times, the space is still the prime ideals of the ordinary quotient $R/(f)$. As far as the space alone is concerned, you don't see the extra data yet.

Here's where that extra data does play a role. The extra isomorphisms floating around allow you to calculate homotopy groups. All these extra isomorphisms can produce nontrivial loops in your category, and by taking homotopy groups, you can detect if the category you're working with has those extra isomorphisms that ordinary rings won't have.

The extra data actually turns the "sheaf of rings" into a "sheaf of categorical rings[6]." Basically while the space itself doesn't change, the extra data we carry around on top of it does. It's worth pointing out though, even if I'm not giving details here, that for categorical rings, their version of a tensor product doesn't have the same issues about killing information — in fact, their tensor product is exactly the derived tensor product repair we mentioned earlier.

So finally (and for real this time, at least for this talk):

Theorem 8 (Bézout's Theorem)**.** *If $\deg f = m$ and $\deg g = n$, then over $\mathbf{P}$ if you construct categorical rings and take derived tensor products, you'll be able to see that the intersection number of f and g is $m \cdot n$.*

§A Appendix

The two puns in the title are the need for a point at ∞ to build $\mathbf{P}$, and those "categorical rings" are actually ∞-groupoids / ∞-categories.

This talk was aimed at an audience of general grad students, with particular focus to first year students who may have had no algebraic geometry background. So certain topics are over-simplified or only given intution in lieu of rigor — please keep this in mind as you read these notes. If nothing else, consider this a great place to find the words you need to google in order to learn more! ☺

[5]Of course, imposing the relation $f = 0$ is also going to start connecting other objects too, and the picture ought to reflect that. Let's not get into the weeds though.

[6]Reminder: I've only said the words "scheme" and "sheaf" in passing, so you aren't expected to know what this means.